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ON THE PROBLEM OF STEADY VIBRATIONS OF A PLANE WITH A SLIT

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An asymptotic method permitting us to obtain the solution in the form of simple formulas for large values of some characteristic dimensionless parameter is used to investigate the problem of steady vibrations of a plane with a finite slit. Computational formulas are presented to determine the function characterizing the displacement of points of the slit edges, and the critical value of stress. A numerical investigation of the results obtained is presented.

The plane problem of elasticity theory concerning the steady vibrations of a plane with a slit of length $2a$ is considered. a normal load $\sigma_y = -q_0 + q_1 \cos(\omega t)$ (where $q_0 > q_1 > 0$, t is the time) is applied to the slit edges for $|x| \leqslant a$, $y = 0$. This problem has been investigated earlier in [1]. An asymptotic method developed in [2] is used below for the solution. This method permits obtaining the solution of the problem under consideration in the form of simple formulas for large values of the parameter $\lambda = (G/\rho)^{1/2}(\omega a)^{-1}$ (where ρ is the density, and G is the shear modulus of the plane).

Let us seek the function $u_y(x, 0)$, $|x| \leqslant a$ characterizing the displacement of points of the slit edges in the following form:

$$u_y = (1-\nu) G^{-1} q_0 \sqrt{a^2 - x^2} + \operatorname{Re}\{\gamma(x) e^{-i\omega t}\} \tag{1}$$

Here ν is the Poisson's ratio, and the first member, corresponding to the case $q_1 = 0$, is the known solution of the Griffith problem for a plane with a crack of length $2a$ to whose edges a normal load $\sigma_y = -q_0$ is applied. The following integral equation can be obtained to determine the function $\gamma(x)$ by using operational calculus methods (the kernel $Q(h)$ is understood in the sense of generalized functions):

$$\int_{-1}^{1} \varphi(z) Q\left(\frac{z-r}{\lambda}\right) dz = \pi \frac{(1-\nu) q_1 \lambda}{G}, \quad |r| < 1 \tag{2}$$

$$r = x/a, \qquad \varphi(r) = \gamma'(x) \tag{3}$$

$$Q(h) = \int_0^\infty L(u) \sin(uh)\, du \qquad \left(h = \frac{z-r}{\lambda}\right) \tag{4}$$

$$L(u) = \frac{(2u^2-1)^2 - 4u^2\sqrt{u^2-1}\sqrt{u^2-\varepsilon^2}}{2(\varepsilon^2-1)u\sqrt{u^2-\varepsilon^2}}, \qquad \varepsilon = \sqrt{\frac{1-2\nu}{2(1-\nu)}} \tag{5}$$

Using the integral representation and recursion formulas for the Hankel function $H_k^{(2)}(z)$ [3], the kernel (4) can be represented as follows:

$$Q(h) = \frac{\pi i}{\varepsilon^2 - 1}\left\{\left(\frac{2\varepsilon}{h^2} - \varepsilon^3 + \varepsilon\right) H_1^{(2)}(\varepsilon h) - \right. \tag{6}$$

$$\frac{2}{h^2} H_1^{(2)}(h) - \frac{1}{h}\left[\varepsilon^2 H_0^{(2)}(\varepsilon h) - H_0^{(2)}(h)\right] - \frac{1}{4}\int H_0^{(2)}(\varepsilon h)\, dh\Big\}$$

Taking account of the series expansions of cylinder functions [3], we find from (6) ($c_* = C - \ln 2$, where C is the Euler constant [3]):

$$Q(h) = \frac{1}{h} + \sum_{n=0}^{\infty} b_n h^{2n+1} + \ln|h| \sum_{n=0}^{\infty} d_n h^{2n+1} \tag{7}$$

$$b_0 = \frac{1}{\varepsilon^2 - 1}\left[\varepsilon^4\left(-\frac{3}{4}\ln\varepsilon - \frac{3}{4}c_* + \frac{5}{16}\right) + \varepsilon^2\left(\ln\varepsilon + c_* - \frac{1}{2}\right) - \frac{3}{4}c_* - \frac{1}{2}\ln\varepsilon + \frac{11}{16} + \pi i\left(-\frac{3}{8}\varepsilon^4 + \frac{1}{2}\varepsilon^2 - \frac{3}{8}\right)\right]$$

$$b_1 = \frac{1}{\varepsilon^2 - 1}\left[\varepsilon^6\left(\frac{5}{48}\ln\varepsilon + \frac{5}{48}c_* - \frac{73}{576}\right) + \varepsilon^4\left(-\frac{1}{8}\ln\varepsilon - \frac{1}{8}c_* + \frac{5}{32}\right) + \varepsilon^2\left(\frac{1}{24}\ln\varepsilon + \frac{1}{24}c_* - \frac{1}{18}\right) + \frac{1}{48}c_* - \frac{17}{576} + \pi i\left(\frac{5}{96}\varepsilon^6 - \frac{1}{16}\varepsilon^4 + \frac{1}{48}\varepsilon^2 + \frac{1}{96}\right)\right]$$

$$d_0 = \frac{1}{\varepsilon^2 - 1}\left(-\frac{3}{4}\varepsilon^4 + \varepsilon^2 - \frac{3}{4}\right),$$

$$d_1 = \frac{1}{\varepsilon^2 - 1}\left(\frac{5}{48}\varepsilon^6 - \frac{1}{8}\varepsilon^4 + \frac{1}{24}\varepsilon^2 + \frac{1}{48}\right) \quad \text{etc.}$$

Let us insert $Q(h)$ in the form (7) into (2), and after transformations we obtain

$$\int_{-1}^{1} \frac{\varphi(z)}{z - r}\, dz = \pi f(r), \qquad |r| < 1 \tag{8}$$

$$f(r) = \frac{(1-\nu)q_1}{G} - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{\lambda^{2n+2}}\int_{-1}^{1}\varphi(z)(z-r)^{2n+1} \times \left[b_n + d_n \ln\frac{|z-r|}{\lambda}\right] dz$$

Applying the inversion formula to the singular integral equation (8), we obtain an integral equation of the second kind in the function $\varphi(r)$

$$\varphi(r) = \frac{1}{\pi\sqrt{1-r^2}}\left[P + \frac{(1-\nu)\pi q_1}{G} r + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{\lambda^{2n+2}}\int_{-1}^{1}\frac{\sqrt{1-z^2}}{z-r}\, dz \int_{-1}^{1}\varphi(\tau)(\tau - z)^{2n+1}\left(b_n + d_n \ln\frac{|\tau - z|}{\lambda}\right) d\tau\right] \tag{9}$$

$$P = \int_{-1}^{1}\varphi(r)\, dr \tag{10}$$

Taking account of (1), (3) and the obvious condition $u_y(\pm a, 0) = 0$, it is easy to see that P equals zero. Let us seek the solution of (9) as follows:

$$\varphi(r) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi_{nm}(r)\lambda^{-2n}\ln^m\lambda \tag{11}$$

Inserting $\varphi(r)$ in the form (11) into the left and right sides of (9) and equating the expressions for identical powers of λ and $\ln\lambda$ on both sides of the relationship obtained by this means, we arrive at the following infinite system of integral equations to determine the functions $\varphi_{nm}(r)$:

$$\varphi_{00}(r) = \frac{(1-\nu)\,q_1}{G}\,\frac{r}{\sqrt{1-r^2}} \tag{12}$$

$$\varphi_{10}(r) = \frac{1}{\pi^2\sqrt{1-r^2}}\int_{-1}^{1}\frac{\sqrt{1-z^2}}{z-r}\,dz\int_{-1}^{1}\varphi_{00}(\tau)(\tau-z)\,[b_0+d_0\ln|\tau-z|]\,d\tau$$

$$\varphi_{11}(r) = -\frac{d_0}{\pi^2\sqrt{1-r^2}}\int_{-1}^{1}\frac{\sqrt{1-z^2}}{z-r}\,dz\int_{-1}^{1}\varphi_{00}(\tau)(\tau-z)\,d\tau \qquad \text{etc.}$$

Omitting the intermediate computations on (12), let us present the expression determining the function $\varphi(r)$ (the calculations were performed for $\nu = 0.3$)

$$\varphi(r) = \frac{(1-\nu)\,q_1}{G}\,\frac{r}{\sqrt{1-r^2}}\,[\Omega_1(r) + i\Omega_2(r)] \tag{13}$$

$$\Omega_1(r) = 1 - \lambda^{-2}(0.3679\,r^2 - 0.8175 - 0.3679\ln\lambda) - \lambda^{-4}\,(-0.03129\,r^4 + 0.3014\,r^2 - 0.1337 - \ln\lambda\,(-0.1877\,r^2 + 0.5131) - 0.1353\ln^2\lambda) + O(\lambda^{-6}\ln^3\lambda)$$

$$\Omega_2(r) = -0.5778\,\lambda^{-2} - \lambda^{-4}\,(-0.2949\,r^2 - 0.8060 + 0.4251\ln\lambda) + O(\lambda^{-6}\ln^3\lambda)$$

From (13) and (3) we obtain

$$\gamma'(x) = \frac{(1-\nu)\,q_1 x}{G\sqrt{a^2-x^2}}\left[\Omega_1\left(\frac{x}{a}\right) + i\Omega_2\left(\frac{x}{a}\right)\right] \tag{14}$$

The connection between the length of the crack and the load applied to the slit edges is determined by the formula [1]

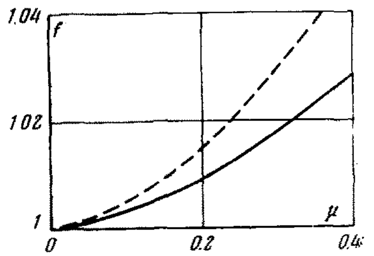

Fig. 1

$$K_I = K_{Ic} \tag{15}$$

$$K_I = \lim_{x\to a+0}\sqrt{2\pi(x-a)}\,\sigma_y(x,0) = \tag{16}$$

$$-\lim_{x\to a-0}\frac{G\sqrt{2\pi(a-x)}}{1-\nu}\,\frac{\partial}{\partial x}\,u_y(x,0)$$

Here K_{Ic} is the critical stress intensity factor. Inserting u_y in the form (1) and taking account of (14) and (16), we obtain

$$K_I = \sqrt{a\pi}\,q_0\,\{1 - \varkappa\,[\Omega_1(1)\cos\omega t + \Omega_2(1)\sin\omega t]\} \tag{17}$$

Substituting the maximum value of K_I into (15), we find

$$q_0\,(1+\varkappa\Omega_*) = K_{Ic}\,(a\pi)^{-1/2} \tag{18}$$

$$\varkappa = q_1/q_0, \qquad \Omega_* = [\Omega_1^2(1) + \Omega_2^2(1)]^{1/2}$$

From (1) and (14) we determine the function $u_y(x, 0)$ for $|x| \leqslant a$

$$u_y(x,0) = (1-\nu)\,q_0 G^{-1}\sqrt{a^2-x^2}\,(1-\varkappa)\left[\Omega_3\left(\frac{x}{a}\right)\cos\omega t + \Omega_4\left(\frac{x}{a}\right)\sin\omega t\right] \tag{19}$$

$$\Omega_3(r) = 1 - \lambda^{-2}(0.1226\,r^2 - 0.5723 - 0.3679\ln\lambda) - \lambda^{-4}\,(-0.006258\,r^4 + (0.09212 + 0.06258\ln\lambda)\,r^2 + 0.05052 - 0.3879\ln\lambda - 0.1353\ln^2\lambda) + O(\lambda^{-6}\ln^3\lambda)$$

$$\Omega_4(r) = -0.5778\,\lambda^{-2} - \lambda^{-4}(-0.09830\,r^2 + 0.6093 + 0.4251\ln\lambda) + O(\lambda^{-6}\ln^3\lambda)$$

We obtain the function characterizing the maximum displacements of points of the slit edges $u_*(x)$ from (19)

$$u_*(x) = (1-\nu)\,q_0 G^{-1}\sqrt{a^2 - x^2}\,\{1 + \varkappa\,[\Omega_3^2(x/a) + \Omega_4^2(x/a)]^{1/2}\} \qquad (20)$$

As $\omega \to 0$ ($\lambda \to \infty$) we obtain the solution of the corresponding static problem from (18) and (20).

As computations have shown, the formulas (17)–(20) obtained can be used in practice for $2 \leqslant \lambda < \infty$. The solid curves presented in Figs. 1–3 correspond to the value $\varkappa = 0.25$, and the dashes correspond to $\varkappa = 0.5$. A graph of the function

$$f(\mu) = \frac{1 + \Omega_*\varkappa}{1+\varkappa} = \frac{K_{Ic}}{q_0(1+\varkappa)\sqrt{a\pi}} \qquad \left(\mu = \frac{\omega a}{c_1},\quad c_1 = \left(\frac{G}{\rho}\right)^{1/2}\right)$$

is presented in Fig. 1.

The dependence of the quantity q_0 / K_{Ic} on half the slit length a is presented in Fig. 2. Values of the quantity a are given in cm, while the values of q_0 / K_{Ic} are given in $(\mathrm{cm})^{-1/2}$. The upper solid and the upper dashed curves correspond to the static case $\omega = 0$. The remaining curves correspond to the frequency $8\times10^4\ \mathrm{sec}^{-1}$. For $a < 0.5$ all the solid and dashed curves practically agree with the corresonding curves for $\omega = 0$. The dependence $U = [(1-\nu)\,q_0\,G^{-1}a]^{-1}\,u_*(x)$ is presented in Fig. 3 for $\mu = 0$ (lower solid and lower dashed curves) and $\mu = 0{,}5$ (remaining curves). It was assumed in the computations that $c_1 = 3.2\times10^3$ m/sec.

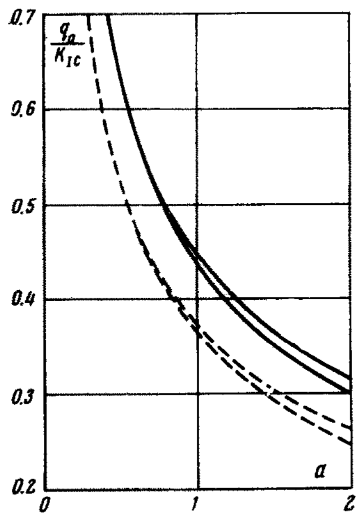

Fig. 2

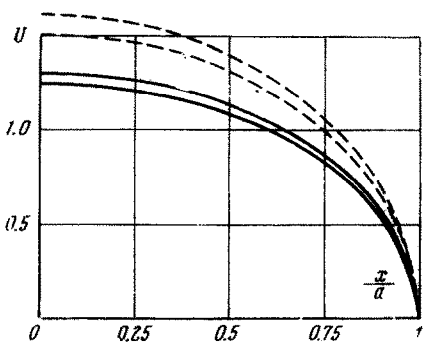

Fig. 3

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